Agreement between the theoretical dependences obtained, the results of numerical computations, and the experimental data indicates the effectiveness of the asymptotic method developed.

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SPECTRAL STRUCTURE OF TURBULENT CONVECTION

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A central problem in the theory of evolution of strong turbulence is, as is well known, the determination of the spectrum of turbulence. Contemporary ideas on scale-invariant spectra are based on Kolmogorov's ideas, introducing the hypothesis of the self-similar nature of the spectrum in an inertial interval and the locality of turbulence [1]. For a long time similarity methods were essentially the only means of theoretical analysis for determining the spectral structure. However, due to the intermittent nature of turbulence dimensionality arguments often do not finally permit finding the form of the spectrum [2], therefore there have recently been numerous attempts at solving the problem of the Kolmogorov spectrum by starting directly from the equations of hydrodynamics.

The increasing interest in self-similar spectra is obviously related to two circumstances. First, the theory of scale-invariant spectra in phase transition problems has been substantially developed lately. Thus, the renormalized-group approach and consideration of problems in arbitrary dimensionality have been powerful means of studying critical effects [3, 4]; these ideas have by now been successfully transferred to strong turbulence [5, 6]. Secondly, the method of conformal mappings [7, 8], first suggested in [9] (see also the review [10]) for finding exact power-law solutions in the theory of weak turbulence, is quite fruitful in solving problems of the Kolmogorov spectrum.

So far all results on the spectra of strong turbulence referred to the case of an isotropic medium.* In reality the effect of anisotropy, related, for example, to the action of gravity forces, is important. In the present paper we solve the problem of finding anisotropic spectra of turbulent convection (the exceptional direction is the vertical).

The effect of convection plays a large role in many physical processes. For example, convective effects underlie a whole variety of solar phenomena [12]; convection is one of the

*Within weak turbulence anisotropic spectra were discussed in [11].

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basic factors forming the structure of the active layer in the ocean [13]; and convective processes affect significantly the dynamics of atmospheric motions [14]. As a rule, in all these phenomena convection is turbulent, and in this connection the problem of the spectrum of turbulent convection is important. In the present work we find the anisotropic spectra of turbulent kinetic energy and the temperature pulsations corresponding to a constant heat flow, and we prove the locality properties of the distributions obtained.

To describe convection in an infinite layer of a viscous incompressible liquid we use the dimensionless equations of motion in the Boussinesq approximation [15] for the velocity field v and the temperature deviation T from the state of hydrostatic equilibrium $T_0 = -Az + T_1$ (A is the equilibrium gradient, and T_1 is the temperature at lowest boundary of the layer):

$$\mathrm{Pr}^{-1} d\mathbf{v}/dt = -\nabla p + \Delta \mathbf{v} + \mathrm{Ra}\, T \cdot \mathbf{e}_z; \tag{1}$$

$$\partial T/\partial t + (\mathbf{v}\nabla) T = \Delta T + \mathbf{v} \cdot \mathbf{e}_z; \tag{2}$$

$$\mathbf{div}\, \mathbf{v} = 0, \tag{3}$$

where $\mathrm{Ra} = g\hat{\beta} A h^4/\nu\chi$ is the Rayleigh number, $\mathrm{Pr} = \nu/\chi$ is the Prandtl number, $\mathbf{e}_z$ is a unit vector along the z-axis, $\hat{\beta}$ is the thermal expansion coefficient, ν and χ are the coefficients of viscosity and thermal conductivity, and h is the layer width. In these equations time is measured in units of h^2/ν, velocity in units of χ/h, temperature in units of Ah, and pressure in units of $\rho_0 \nu\chi/h^2$ (ρ_0 is the unperturbed density).

We use the Rayleigh boundary conditions:

$$w = 0,\ \partial \mathbf{v}_\perp/\partial z = 0,\ T = 0 \text{ at } z = 0 \text{ and } z = 1. \tag{4}$$

The temperature field $T(\mathbf{r}_\perp, z, t)$ can be represented as a sum of the field $\overline{T}(z, t)$ averaged over the horizontal field and a deviation from this averaged value $\Theta(\mathbf{r}_\perp, z, t)$:

$$T(\mathbf{r}_\perp, z, t) = \overline{T}(z, t) + \Theta(\mathbf{r}_\perp, z, t)$$

(the velocity field has a vanishing average component).

From Eq. (2) it follows for $\overline{T}$ and Θ that

$$\frac{\partial \Theta}{\partial t} - \Delta\Theta = \frac{\partial \overline{T}}{\partial z} w - \nabla\left(\mathbf{v}\Theta - (\overline{\mathbf{v}\Theta})\right); \tag{2a}$$

$$\frac{\partial \overline{T}}{\partial t} - \frac{\partial^2 \overline{T}}{\partial z^2} = -\frac{\partial}{\partial z}(\overline{w\Theta}), \tag{2b}$$

where w is the vertical velocity component, and the bar denotes horizontal averaging. In what follows we assume that the condition $\mathrm{Pr} \gg 1$ is satisfied, in which case the inertial term in Eq. (1) can be neglected and the temperature fluctuation can be related to the velocity field

$$-\nu\Delta\mathbf{v} = g\hat{\beta} T - \frac{1}{\rho_0}\nabla p. \tag{5}$$

We seek a solution of system (1)-(3) with boundary conditions (4) in the form of an expansion in eigenfunctions of the linear boundary-value problem:

$$\begin{gathered} w(\mathbf{r}_\perp, z, t) = \sum_{k_z=-\infty}^{\infty} \int w_{\mathbf{k}_\perp}^{k_z}(t)\, e^{i\mathbf{k}_\perp \mathbf{r}_\perp + ik_z z} d\mathbf{k}_\perp, \\ T(\mathbf{r}_\perp, z, t) = \sum_{k_z=-\infty}^{\infty} \int T_{\mathbf{k}_\perp}^{k_z}(t)\, e^{i\mathbf{k}_\perp \mathbf{r}_\perp + ik_z z} d\mathbf{k}_\perp, \\ \mathbf{u}_\perp(\mathbf{r}_\perp, z, t) = -\sum_{k_z=-\infty}^{\infty} \frac{\mathbf{k}_\perp k_z}{k_\perp^2} w_{\mathbf{k}_\perp}^{k_z}(t)\, e^{i\mathbf{k}_\perp \mathbf{r}_\perp + ik_z z} d\mathbf{k}_\perp, \\ w_{\mathbf{k}_\perp}^{k_z} = w_{-\mathbf{k}_\perp}^{*k_z},\ T_{\mathbf{k}_\perp}^{k_z} = T_{-\mathbf{k}_\perp}^{*k_z},\ w_{\mathbf{k}_\perp}^{k_z} = -w_{\mathbf{k}_\perp}^{-k_z},\ T_{\mathbf{k}_\perp}^{k_z} = -T_{\mathbf{k}_\perp}^{-k_z}, \end{gathered} \tag{6}$$

where $\mathbf{k}_\perp$ is the wave vector in the horizontal plane, and $k_z = \pi n/h$ is the quantized magnitude of the z-component momentum (n is a discrete number, characterizing the number of half-waves stacked vertically).

It follows from (5) that

$$w_{\mathbf{k}_\perp}^{k_z} = \frac{g\widehat{\beta}}{\nu}\frac{k_\perp^2}{k^4}T_{\mathbf{k}_\perp}^{k_z}, \quad \mathbf{u}_{\perp\mathbf{k}_\perp}^{k_z} = -\frac{g\widehat{\beta}}{\nu}\frac{\mathbf{k}_\perp k_z}{k^4}T_{\mathbf{k}_\perp}^{k_z}, \quad k^2 = k_\perp^2 + k_z^2. \tag{7}$$

Using (7), we obtain for the Fourier component of the temperature field the following equation

$$\frac{\partial T_{\mathbf{k}_\perp}^{k_z}}{\partial t} = \gamma_{\mathbf{k}_\perp}^{k_z} T_{\mathbf{k}_\perp}^{k_z} + \frac{i}{2}\sum_{k_z+k_{z_1}+k_{z_2}=0}\int V_{\mathbf{k}_\perp}^{k_z}\Big|_{\mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}}^{k_{z_1}k_{z_2}} \overset{*}{T}{}_{\mathbf{k}_{\perp_1}}^{k_{z_1}} \overset{*}{T}{}_{\mathbf{k}_{\perp_2}}^{k_{z_2}}\,\delta(\mathbf{k}_\perp + \mathbf{k}_{\perp_1} + \mathbf{k}_{\perp_2})\,d\mathbf{k}_{\perp_1}d\mathbf{k}_{\perp_2}, \tag{8}$$

where

$$\gamma_{\mathbf{k}_\perp}^{k_z} = \frac{\chi k_\perp^2}{(kh)^4}(\mathrm{Ra} - \mathrm{Ra}_c); \quad \mathrm{Ra}_c = \frac{h^4 k^6}{k_\perp^2};$$

$$V_{\mathbf{k}_\perp}^{k_z}\Big|_{\mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}}^{k_{z_1}k_{z_2}} = \frac{g\widehat{\beta}}{\nu}\left[\frac{k_{\perp_1}^2}{k_1^4}\left(k_{z_2} + \frac{1}{2}k_{z_1} - \frac{1}{2}k_{z_1}\left(\frac{k_{\perp_2}^2 - k_\perp^2}{k_{\perp_2}^2}\right)\right) + \frac{k_{\perp_2}^2}{k_2^4}\left(k_{z_1} + \frac{1}{2}k_{z_2} - \frac{1}{2}k_{z_2}\left(\frac{k_{\perp_1}^2 - k_\perp^2}{k_{\perp_2}^2}\right)\right)\right].$$

The matrix element of the interaction $V_{\mathbf{k}_\perp}^{k_z}\Big|_{\mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}}^{k_{z_1}k_{z_2}}$ possesses the symmetry $V_{\mathbf{k}_\perp}^{k_z}\Big|_{\mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}}^{k_{z_1}k_{z_2}} = V_{\mathbf{k}_\perp}^{k_z}\Big|_{\mathbf{k}_{\perp_2}\mathbf{k}_{\perp_1}}^{k_{z_2}k_{z_1}}$ and satisfies the Jacobi identity

$$V_{\mathbf{k}_\perp}^{k_z}\Big|_{\mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}}^{k_{z_1}k_{z_2}} + V_{\mathbf{k}_{\perp_1}}^{k_{z_1}}\Big|_{\mathbf{k}_{\perp_2}\mathbf{k}_\perp}^{k_{z_2}k_z} + V_{\mathbf{k}_{\perp_2}}^{k_{z_2}}\Big|_{\mathbf{k}_\perp\mathbf{k}_{\perp_1}}^{k_z k_{z_1}} = 0, \tag{9}$$

being a consequence of the conservation law of heat flux.

In what follows we will need the asymptotic matrix element in the two limiting cases: $k_z \gg k_\perp$ and $k_z \ll k_\perp$. In this case $V_{\mathbf{k}_\perp}^{k_z}\Big|_{\mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}}^{k_{z_1}k_{z_2}}$ is a homogeneous function of its arguments

$$V_{\lambda\mathbf{k}_\perp}^{\xi k_z}\Big|_{\lambda\mathbf{k}_{\perp_1}\lambda\mathbf{k}_{\perp_2}}^{\xi k_{z_1}\xi k_{z_2}} = \lambda^t \xi^r V_{\mathbf{k}_\perp}^{k_z}\Big|_{\mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}}^{k_{z_1}k_{z_2}},$$

while for $k_z \gg k_\perp$, $t = 2$, $r = -3$, and for $k_z \ll k_\perp$ $t = -2$, $r = 1$.

A specific problem is the presence of a mean temperature field, described by a vanishing spatial harmonic in (8).

The presence of an average component complicates the statistical treatment within Eq. (8), therefore it is more convenient to reformulate the problem in terms of the vertical velocity component, having a vanishing mean value:

$$\frac{\partial w_{\mathbf{k}_\perp}^{k_z}}{\partial t} = \gamma_{\mathbf{k}_\perp}^{k_z} w_{\mathbf{k}_\perp}^{k_z} + \frac{i}{2}\sum_{k_z+k_{z_1}+k_{z_2}=0}\int \frac{k_\perp^2}{k^4}\frac{k_1^4 k_2^4}{k_{\perp_1}^2 k_{\perp_2}^2} V_{\mathbf{k}_\perp}^{k_z}\Big|_{\mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}}^{k_{z_1}k_{z_2}} \overset{*}{w}{}_{\mathbf{k}_{\perp_1}}^{\mathbf{k}_{z_1}} \overset{*}{w}{}_{\mathbf{k}_{\perp_2}}^{\mathbf{k}_{z_2}} \times$$
$$\times\,\delta(\mathbf{k}_\perp + \mathbf{k}_{\perp_1} + \mathbf{k}_{\perp_2})\,d\mathbf{k}_{\perp_1}d\mathbf{k}_{\perp_2}. \tag{10}$$

We are interested in the evolution regime of turbulent convection. As indicated by experiments (see, e.g., [16]), for Pr > 5 this regime occurs for Ra $5 \cdot 10^5$. First we consider qualitatively the structure of turbulent convection. The heat excess in the lower layer is transferred by the motion of vortices of large sizes. These vortices exist for a time not longer than the time of liquid motion in the neighborhood of vortices, and therefore they do not succeed in carrying up the heat excess to the upper boundary of the layer. Large-scale vortices decay more finely, and, thus, the whole region of turbulent convection consists of an ensemble of vortices of different scales. Since vortices are subject to the action of gravity forces, turbulence has an anisotropic nature (with the exceptional direction being the vertical). To describe the turbulence we introduce the following characteristics:

the spectral kinetic energy density

$$F(\mathbf{k}, t) = \left\langle \mathbf{u}_{\perp\mathbf{k}_\perp}^{k_z} \mathbf{u}_{\perp\mathbf{k}_\perp}^{*k_z} + w_{\mathbf{k}_\perp}^{k_z} w_{\mathbf{k}_\perp}^{*k_z} \right\rangle$$

and the spectral temperature pulsation density

$$F_T(\mathbf{k}, t) = \left\langle \Theta_{\mathbf{k}_\perp}^{k_z} \Theta_{\mathbf{k}_\perp}^{*k_z} \right\rangle,$$

where < > is the average over the statistical ensemble.

Taking into account (7), (6), the following relations can be obtained for F(k, t) and F_T(k, t):

$$F(\mathbf{k}, t) = \frac{k^2}{k_\perp^2} \left\langle w_{\mathbf{k}_\perp}^{k_z} w_{\mathbf{k}_\perp}^{*k_z} \right\rangle \equiv \frac{k^2}{k_\perp^2} I_{\mathbf{k}_\perp}^{k_z},$$

$$F_T(\mathbf{k}, t) = \left(\frac{\nu k^4}{g\hat{\beta} k_\perp^2}\right)^2 \left\langle w_{\mathbf{k}_\perp}^{k_z} w_{\mathbf{k}_\perp}^{*k_z} \right\rangle \equiv \left(\frac{\nu k^4}{g\hat{\beta} k_\perp^2}\right)^2 I_{\mathbf{k}_\perp}^{k_z}.$$

Thus, knowledge of the quantity $I_{\mathbf{k}_\perp}^{k_z} = \left\langle w_{\mathbf{k}_\perp}^{k_z} w_{\mathbf{k}_\perp}^{*k_z} \right\rangle$ makes it possible to determine the required characteristics. From Eq. (10) it follows for $I_{\mathbf{k}_\perp}^{k_z}$ that

$$\frac{\partial I_{\mathbf{k}_\perp}^{k_z}}{\partial t} = 2\gamma_{\mathbf{k}_\perp}^{k_z} I_{\mathbf{k}_\perp}^{k_z} + \frac{1}{2}\,\mathrm{Im} \sum_{k_z + k_{z_1} + k_{z_2} = 0} \int \frac{k_\perp^2 k_1^4 k_2^4}{k^4 k_{\perp_1}^2 k_{\perp_2}^2} V_{\mathbf{k}_\perp}^{k_z}\Big|_{\mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}}^{k_{z_1} k_{z_2}} J_{\mathbf{k}_\perp \mathbf{k}_{\perp_1} \mathbf{k}_{\perp_2}}^{k_z k_{z_1} k_{z_2}} \delta(\mathbf{k}_\perp + \mathbf{k}_{\perp_1} + \mathbf{k}_{\perp_2})\, d\mathbf{k}_{\perp_1} d\mathbf{k}_{\perp_2}, \quad (11)$$

where

$$J_{\mathbf{k}_\perp \mathbf{k}_{\perp_1} \mathbf{k}_{\perp_2}}^{k_z k_{z_1} k_{z_2}} = \left\langle w_{\mathbf{k}_\perp}^{k_z} w_{\mathbf{k}_{\perp_1}}^{k_{z_1}} w_{\mathbf{k}_{\perp_2}}^{k_{z_2}} \right\rangle.$$

To study the statistical characteristics of Eq. (10) it is convenient to use Wyld's diagram technique [17], using two quantities, the spectral density $I_{\mathbf{k}_\perp \omega}^{k_z}$ and the generalized Green's function $G_{\mathbf{k}_\perp \omega}^{k_z}$. First-order perturbation theory corresponds to Kraichnan's model [18] of direct interactions.* As shown in [22], however, this approximation enhances the effect of large-scale motions on the evolution of small-scale inhomogeneities and leads to a spectrum in disagreement with the Kolmogorov spectrum, which was quite well verified experimentally [2].

Some of the most diverging diagrams, describing transport, were summed in [8], while the improved equations of direct interactions already contain solutions with the Kolmogorov index values [23].

For the case under consideration the improved equations in the k-ω representation are

$$\begin{gathered} I_{\mathbf{k}_\perp \omega}^{k_z} = \left| G_{\mathbf{k}_\perp \omega}^{k_z} \right|^2 \Phi_{\mathbf{k}_\perp \omega}^{k_z}, \\ G_{\mathbf{k}_\perp \omega}^{k_z} = \left(\omega - i\gamma_{\mathbf{k}_\perp}^{k_z} - \Sigma_{\mathbf{k}_\perp \omega}^{k_z}\right)^{-1}, \\ \Phi_{\mathbf{k}_\perp \omega}^{k_z} = \sum_{\mathbf{k}_{z_1}, \mathbf{k}_{z_2}} \int dq_1 dq_2 \left[\tilde{V}_{\mathbf{k}_\perp}^{k_z}\Big|_{\mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}}^{k_{z_1} k_{z_2}}\right]^2 I_{q_1}^{k_{z_1}} I_{q_2}^{k_{z_2}} \times \\ \times \left[\delta(q + q_1 + q_2)\, \delta_{k_z, -(k_{z_1} + k_{z_2})} - \delta(q + q_1)\, \delta_{k_z, -k_{z_1}}\right], \end{gathered} \quad (12)$$

*As applied to the whole system (1), (2a), (2b), the equations of direct interactions were formulated in [19-21].

$$\Sigma^{k_z}_{\mathbf{k}_\perp\omega} = \sum_{k_{z_1},\, k_{z_2}} \int dq_1 dq_2 \widetilde{V}^{k_z}_{\mathbf{k}_\perp}\Big|^{k_{z_1}k_{z_2}}_{\mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}} \widetilde{V}^{k_{z_1}}_{\mathbf{k}_{\perp_1}}\Big|^{k_z k_{z_2}}_{\mathbf{k}_\perp \mathbf{k}_{\perp_2}} I^{k_{z_1}}_{q_1} G^{k_{z_2}}_{q_2} \left[\frac{1}{2}\,\delta(q+q_1+q_2)\,\delta_{k_z,-(k_{z_1}+k_{z_2})} - \delta(q+q_1)\,\delta_{k_z,-k_{z_1}}\right],$$

where

$$q = (\mathbf{k}_\perp\omega); \quad \widetilde{V}^{k_z}_{\mathbf{k}_\perp}\Big|^{k_{z_1}k_{z_2}}_{\mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}} = \frac{k_\perp^2 k_1^4 k_2^4}{k^4 k^2_{\perp_1} k^2_{\perp_2}}\, V^{k_z}_{\mathbf{k}_\perp}\Big|^{k_{z_1}k_{z_2}}_{\mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}}.$$

We further express the triple correlator $J^{k_z k_{z_1} k_{z_2}}_{\mathbf{k}_\perp \mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}}$ in the form of a power series in $I^{k_z}_q$ and $G^{k_z}_q$. In this case Eq. (11) is rewritten within the direct interaction model in the form

$$\frac{\partial I^{k_z}_{\mathbf{k}_\perp}}{\partial t} = 2\gamma^{k_z}_{\mathbf{k}_\perp} I^{k_z}_{\mathbf{k}_\perp} + \frac{1}{2}\,\mathrm{Im} \sum_{k_z+k_{z_1}+k_{z_2}=0} \int dq_1 dq_2 d\omega \widetilde{V}^{k_z}_{\mathbf{k}_\perp}\Big|^{k_{z_1}k_{z_2}}_{\mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}} \times$$
$$\times \left\{\widetilde{V}^{k_z}_{\mathbf{k}_\perp}\Big|^{k_{z_1}k_{z_2}}_{\mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}} G^{k_z}_q I^{k_{z_1}}_{q_1} I^{k_{z_2}}_{q_2} + \widetilde{V}^{k_{z_2}}_{\mathbf{k}_{\perp_2}}\Big|^{k_z k_{z_1}}_{\mathbf{k}_\perp \mathbf{k}_{\perp_1}} G^{k_{z_2}}_{q_2} I^{k_z}_q I^{k_{z_1}}_{q_1} + \widetilde{V}^{k_{z_1}}_{\mathbf{k}_{\perp_1}}\Big|^{k_{z_2}k_z}_{\mathbf{k}_{\perp_2}\mathbf{k}_\perp} G^{k_{z_1}}_{q_1} I^{k_{z_2}}_{q_2} I^{k_z}_q\right\} \delta(q+q_1+q_2)\,\delta_{k_z,-(k_{z_1}+k_{z_2})}. \tag{13}$$

Equation (13) is similar to the kinetic equation for waves in the theory of weak turbulence.

We are interested in solving Eq. (13) in the inertial interval, where the effect of the energy-containing region and the dissipation region can be neglected, and the main contribution to formation of nonequilibrium spectral flows is provided by the collision integral. The applicability limit of this treatment is discussed below.

We initially determine the stationary spectra from dimensionality considerations. For this we use the Kolmogorov hypothesis concerning the heat flow:

$$\varepsilon_T \sim \frac{k_\perp^2 k_z F_T(\mathbf{k})}{\tau_{\mathrm{int}}} = \mathrm{const}, \tag{14}$$

where τ_{int} is the characteristic time of nonlinear temperature pulsation interactions.

From Eq. (8) one can estimate

$$\frac{1}{\tau_{\mathrm{int}}} \sim \left(\frac{g\hat{\beta}}{\nu}\right) F_T(\mathbf{k})\, k_\perp^{t+1} k_z^{r+\frac{1}{2}}. \tag{15}$$

We then obtain from (14), (15)

$$F_T(\mathbf{k}) \sim (\nu\varepsilon_T/g\hat{\beta})^{2/3} k_\perp^{-(2t+6)/3} k_z^{-(2r+3)/3}. \tag{16}$$

Using Eq. (7) for $F(\mathbf{k})$, we have

$$F(\mathbf{k}) \sim \varepsilon_T^{2/3} (g\hat{\beta}/\nu)^{4/3} k_\perp^{-2t/3} k_z^{-(2r+3)/3} k^{-6}. \tag{17}$$

In the limiting cases, when $k_z \gg k_\perp$ or $k_z \ll k_\perp$, Eqs. (16), (17) are, respectively,

$$F_T(\mathbf{k}) \sim k_\perp^{-2/3} k_z^{-5/3}, \quad F(\mathbf{k}) \sim k_\perp^{-14/3} k_z^{-5/3} \quad (k_z \ll k_\perp); \tag{18}$$

$$F_T(\mathbf{k}) \sim k_\perp^{-10/3} k_z^{1}, \quad F(\mathbf{k}) \sim k_\perp^{-4/3} k_z^{-5} \quad (k_z \gg k_\perp). \tag{19}$$

For an arbitrary relation between k_z and $k_\perp$ dimensionality considerations can give one-dimensional spectra (depending on $|\mathbf{k}|$):

$$F_T(\mathbf{k}) \sim k^{-7/3}, \quad F(\mathbf{k}) \sim k^{-19/3}.$$

In this case the nature of the angular distribution, i.e., the nature of spectral ansisotropy, remains undetermined.

We note that the spectra obtained are essentially based on the locality hypothesis.

We determine analytically the stationary spectra of turbulent convection. For this it is necessary to find the accurate solutions of Eqs. (12), (13). Due to the separateness of the vertical direction it is natural to seek a distribution dependent on k_z and $k_\perp$. At the present time only one method is known of finding exact solutions, based on the factorization method [10], while symmetry and homogeneity properties of the interaction matrix elements are widely used. In the given case, however, this method cannot be applied directly, since $V_{\mathbf{k}_\perp}^{k_z}\Big|_{\mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}}^{k_{z_1}k_{z_2}}$ is not a homogeneous function of k_z and $k_\perp$ in the whole wave number interval, and the factorization procedure can be carried out only in the limiting cases, when either $k_z \gg k_\perp$ or $k_z \ll k_\perp$. In these limiting cases we find solutions of Eqs. (12), (13), transforming in advance in them from summation over k_z to integration:

$$\sum_{k_z} \to \frac{h}{(2\pi)} \int dk_z.$$

The solution of these equations is sought in self-similar form

$$G_{\mathbf{k}_\perp\omega}^{k_z} = \frac{1}{k_\perp^s k_z^p}\, g\left(\frac{\omega}{k_\perp^s k_z^p}\right), \quad I_{\mathbf{k}_\perp\omega}^{k_z} = \frac{1}{k_\perp^{s+\alpha} k_z^{p+\beta}}\, f\left(\frac{\omega}{k_\perp^s k_z^p}\right). \tag{20}$$

The first two relations between the unknown indices are obtained from the Dyson equation (12)

$$2s + \alpha = \tilde{t} + 2, \; 2p + \beta = \tilde{r} + 1, \tag{21}$$

where $\tilde{t} = 0$, $\tilde{r} = 1$ both for $k_z \ll k_\perp\; k_z \gg k_\perp$ and for $k_z \gg k_\perp$ ($\tilde{t}$ and $\tilde{r}$ are the homogeneity powers of the matrix element $\tilde{V}_{\mathbf{k}_\perp}^{k_z}\Big|_{\mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}}^{k_{z_1}k_{z_2}}$ in the cases under consideration).

The following two relations between s, α, p, and β are obtained by solving the stationary equations (13)

$$I_{\mathrm{st}} = \operatorname{Im} \int d\omega dq_1 dq_2 dk_{z_1} dk_{z_2} \tilde{V}_{\mathbf{k}_\perp}^{k_z}\Big|_{\mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}}^{k_{z_1}k_{z_2}} \Big\{ \tilde{V}_{\mathbf{k}_\perp}^{k_z}\Big|_{\mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}}^{k_{z_1}k_{z_2}} G_q^{k_z} I_{q_1}^{k_{z_1}} I_{q_2}^{k_{z_2}} + \tag{22}$$

$$+ \tilde{V}_{\mathbf{k}_{\perp_2}}^{k_{z_2}}\Big|_{\mathbf{k}_\perp\mathbf{k}_{\perp_1}}^{k_z k_{z_1}} G_{q_2}^{k_{z_2}} I_q^{k_z} I_{q_1}^{k_{z_1}} + \tilde{V}_{\mathbf{k}_{\perp_1}}^{k_{z_1}}\Big|_{\mathbf{k}_{\perp_2}\mathbf{k}_\perp}^{k_{z_2}k_z} G_{q_1}^{k_{z_1}} I_{q_2}^{k_{z_2}} I_q^{k_z} \Big\} \delta(q + q_1 + q_2)\, \delta(k_z + k_{z_1} + k_{z_2}) = 0.$$

For this we carry out a factorization, introducing a joint conformal transformation in $k_\perp$ and k_z, multiplying in advance the integrand in (22) by k^8/k^4 for symmetrization:

$$k_\perp = k_\perp''\left(\frac{k_\perp}{k_\perp''}\right), \quad k_{\perp_1} = k_\perp'\left(\frac{k_\perp}{k_\perp''}\right), \quad k_{\perp_2} = k_\perp\left(\frac{k_\perp}{k_\perp''}\right), \quad k_z = k_z''\left(\frac{k_z}{k_z''}\right),$$

$$k_{z_1} = k_z'\left(\frac{k_z}{k_z''}\right), \quad k_{z_2} = k_z\left(\frac{k_z}{k_z''}\right), \quad \omega = \omega''\left(\frac{k_\perp}{k_\perp''}\right)^s\left(\frac{k_z}{k_z''}\right)^p.$$

$$\omega_1 = \omega'\left(\frac{k_\perp}{k_\perp''}\right)^s\left(\frac{k_z}{k_z''}\right)^p, \quad \omega_2 = \omega\left(\frac{k_\perp}{k_\perp''}\right)^s\left(\frac{k_z}{k_z''}\right)^p.$$

In this case the second term in (22) transforms on the scale-invariant spectrum (20) into the first with a factor $\left(\frac{k_\perp}{k_{\perp_2}}\right)^x\left(\frac{k_z}{k_{z_2}}\right)^y$, where $x = \tilde{\tilde{t}} + 4 - s - 2\alpha$; $y = \tilde{\tilde{r}} + 2 - p - 2\beta$ while $\tilde{\tilde{t}} = 2$, $\tilde{\tilde{r}} = 1$ for $k_z \ll k_\perp$ and $\tilde{\tilde{t}} = -2$, $\tilde{\tilde{r}} = 5$ for $k_z \gg k_\perp$ ($\tilde{\tilde{t}}$ and $\tilde{\tilde{r}}$ are the homogeneity powers of the quantity $\tilde{\tilde{V}}_{\mathbf{k}_\perp}^{k_z}\Big|_{\mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}}^{k_{z_1}k_{z_2}} = \frac{k_1^4 k_2^4}{k_{\perp_1}^2 k_{\perp_2}^2} V_{\mathbf{k}_\perp}^{k_z}\Big|_{\mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}}^{k_{z_1}k_{z_2}}$ in the cases under consideration). The third term in (22) is similarly transformed.

As a result the integrand function acquires the form

$$\left(\frac{k_1^4 k_2^4}{k_{\perp_1}^2 k_{\perp_2}^2}\right)^2 V_{\mathbf{k}_\perp}^{k_z}\Big|_{\mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}}^{k_{z_1}k_{z_2}} G_q^{k_z} I_{q_1}^{k_{z_1}} I_{q_2}^{k_{z_2}} \Big\{ V_{\mathbf{k}_\perp}^{k_z}\Big|_{\mathbf{k}_{\perp_1}\mathbf{k}_{\perp_2}}^{k_{z_1}k_{z_2}} + \left(\frac{k_\perp}{k_{\perp_2}}\right)^x \left(\frac{k_z}{k_{z_2}}\right)^y V_{\mathbf{k}_{\perp_2}}^{k_{z_2}}\Big|_{\mathbf{k}_\perp \mathbf{k}_{\perp_1}}^{k_z k_{z_1}} +$$

$$+ \left(\frac{k_\perp}{k_{\perp_1}}\right)^x \left(\frac{k_z}{k_{z_1}}\right)^y V_{\mathbf{k}_{\perp_1}}^{k_{z_1}}\Big|_{\mathbf{k}_{\perp_2}\mathbf{k}_\perp}^{k_{z_2}k_z} \Big\} \delta(q + q_1 + q_2)\, \delta(k_z + k_{z_1} + k_{z_2}).$$

The expression in the curly brackets vanishes for $x = y = 0$ by identity (9). The conditions $x = y = 0$ together with relations (21) lead to the following values for the indices:

$$\alpha = 14/3,\ \beta = 5/3,\ s = -4/3,\ p = 2/3 \quad \text{for } k_z \ll k_\perp$$

$$\alpha = -2/3,\ \beta = 7,\ s = 4/3,\ p = -2 \quad \text{for } k_z \gg k_\perp,$$

while for the spatial spectra $F(k)$ and $F_T(k)$ we obtain the solutions (18), (19) found earlier by dimensionality arguments. We stress once more that the factorization procedure applies only to the limiting cases, when either $k_z \ll k_\perp$ or $k_z \gg k_\perp$, as only in this case is the matrix element a bihomogeneous function of its arguments.

For the results obtained to have a physical meaning it is necessary to prove locality of turbulence. The latter implies that the interaction of modes with scales of the same order is much stronger than mode interaction of different scales. Formally the property of locality implies that the integrals in (22) converge on the spectra obtained. We consider initially convergence in the regions $k_{\perp_1} \ll k_\perp$ and $k_{z_1} \ll k_z (k_{\perp_2} \sim k_\perp,\ k_{z_2} \sim k_z)$.

In this case the most dangerous terms (with which most of the divergence is related) are the terms proportional to $I_{\mathbf{k}_{\perp_1}}^{k_{z_1}}$.

For small $k_{\perp_1}$ and k_{z_1}, using property (9) and taking into account that $G_q = -G^*_{-q}$, these terms are collected in the expression

$$\int_0 dk_{z_1} \int_0 \frac{dk_{\perp_1}}{\sqrt{4k_\perp^2 - k_{\perp_1}^2}} \frac{k_1^8}{k_{\perp_1}^4} V_{\mathbf{k}_{\perp_1}}^{k_{z_1}}\Big|_{\mathbf{k}_\perp - \mathbf{k}_\perp}^{k_z - k_z} V_{\mathbf{k}_\perp}^{k_z}\Big|_{\mathbf{k}_{\perp_1}\mathbf{k}_\perp}^{k_{z_1}k_z} I_{\mathbf{k}_{\perp_1}}^{k_{z_1}}. \tag{23}$$

Since the relation $V_{\mathbf{k}_{\perp_1}}^{k_{z_1}}\Big|_{\mathbf{k}_\perp - \mathbf{k}_\perp}^{k_z - k_z} = 0$ is satisfied exactly, this guarantees convergence of the integrals (23) and implies locality of spectra in the region considered.

Consider convergence in the regions $k_{\perp_1}, k_{\perp_2} \gg k_\perp$ and $k_{z_1}, k_{z_2} \gg k_z$. In this case the most dangerous terms are those linear in $I_{\mathbf{k}_{\perp_1}}^{k_{z_1}}$ and $I_{\mathbf{k}_{\perp_2}}^{k_{z_2}}$. We obtain for them integrals of the form

$$\int^\infty dk_{z_1} \int^\infty \frac{dk_{\perp_1}}{\sqrt{4k_{\perp_1}^2 - k_\perp^2}} \frac{k_1^8}{k_{\perp_1}^4} \left[V_{\mathbf{k}_\perp}^{k_z}\Big|_{\mathbf{k}_{\perp_1} - \mathbf{k}_{\perp_1}}^{k_{z_1} - k_{z_1}} \right]^2 I_{\mathbf{k}_{\perp_1}}^{k_{z_1}}.$$

Convergence of integrals at the upper limit is always guaranteed by the fact that $V_{\mathbf{k}_\perp}^{k_z}\Big|_{\mathbf{k}_{\perp_1} - \mathbf{k}_{\perp_1}}^{k_{z_1} - k_{z_1}} \equiv 0$. Thus, the spectra found are local.

We clarify the applicability limits of the solutions obtained. We first turn attention to the fact that, as follows from (5), for $Pr \gg 1$ the inertia forces are negligible and Archimedes forces are statistically counterbalanced by molecular forces. In this case motions of all scales are subject to action of viscosity, and the Kolmogorov portion of the spectrum $F(k) \sim k^{11/3}$ is absent. For sufficiently small scales viscosity appears to have a tendency to isotropy. The minimal scale, starting with which this effect becomes important, can be determined by comparing the characteristic dissipation time due to viscosity in the scale $l\ \tau_\nu \sim l^2/\nu$ with the characteristic time of convective rise of the corresponding scale under the action of the Archimedes force $\tau_{conv} \sim l/v_l$. Using the relation of (5)

$$\nu v_l / l^2 \sim g\hat{\beta} T_l$$

and the relation $\varepsilon_T \sim \frac{T_l^2 v_l}{l} = \text{const}$, one can find from the condition $\tau_\nu \sim \tau_{conv}$

$$l_{cr} \simeq \frac{\nu^{5/8}}{(g\hat{\beta}\varepsilon_T^{1/2})^{1/4}}.$$

This expression can also be obtained directly from dimensionality considerations, if it is assumed that the quantity l_{cr} depends only on the parameters g, $\hat{\beta}$, ν, and ε_T. Thus, for large Pr the anisotropic spectra found (16), (17) will be realized for up to scale l_{cr}. Vortices with scales from l_{cr} to k_{dis}^{-1}, where molecular thermal conductivity becomes important, will be practically isotropic.

In an isotropic interval of scales, solving Eq. (22) averaged over angles, one obtains the following expression for the spatial spectrum of the vertical velocity component:

$$I_k = B\varepsilon_T^{2/3}\left(\frac{g\hat{\beta}}{\nu}\right)^{4/3} k^{-19/3}. \tag{24}$$

In this case the spectra of kinetic energy and temperature pulsations in spherical normalization acquire the form

$$E_T(k) \sim k^{-1/3} \quad \text{and} \quad E(k) \sim k^{-13/3}. \tag{25}$$

Spectral characteristics of this shape were obtained in [24] by numerical solution of semi-empirical equations of energy balance and temperature pulsations. The isotropic spectra (24), (25) are realized up to the scale k_{dis}^{-1}, which can be estimated by considering the problem with a source:

$$2\gamma_k I_k = I_{st}.$$

A solution of type (24) is valid until the collision integral on this spectrum becomes comparable with the damping term (in the region of large k $\gamma_k \sim \chi k^2$)

$$2\chi B k_{dis}^{2-19/3} \sim B^{3/2}\varepsilon_T^{1/3}\left(\frac{g\hat{\beta}}{\nu}\right)^{2/3} k_{dis}^{-7}$$

or

$$k_{dis} \sim \left[\frac{B^{1/2}}{2\chi}(g\hat{\beta}\varepsilon_T^{1/2}/\nu)^{2/3}\right]^{3/8}.$$

For $k > k_{dis}$ the solution drops off quickly. To determine the constant B we use the heat flow conservation law:

$$\int_0^\infty \gamma_k k^4 I_k k^2 dk = 0.$$

We have

$$\gamma_0 k_0^7 \frac{B}{k_0^{19/3}} \sim \chi B \int_0^{k_{dis}} k^{8-19/3} dk = \frac{3}{8}\chi B k_{dis}^{8/3},$$

where γ_0 is the characteristic value of the increment, and k_0 is the characteristic scale of the instability region. Taking $\gamma_0 \sim \chi\, \mathrm{Ra}/\pi^2 h^2$, $k_0 \sim (\pi/h)\sqrt{2}$, we find

$$B \sim \left[\frac{16\,\mathrm{Ra}}{3\pi^2 k_0^{4/3}}\left(\frac{\nu}{\varepsilon_T^{1/2} g\hat{\beta}}\right)^{2/3}\right]^2.$$

The whole treatment is valid if there exists a sufficiently large transparency region, i.e., if $k_{dis} \gg k_0$. Calculating k_{dis} gives the condition

$$Ra \gg 10^4.$$

For turbulent convection this is certainly satisfied.

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